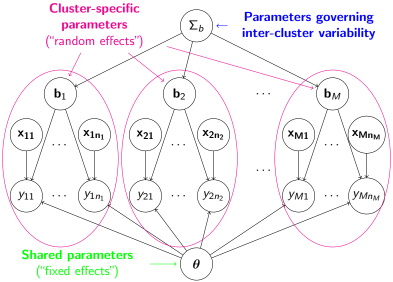
A Brief and Friendly Introduction to Mixed-Effects Models in Linguistics



slides by Roger Levy
presented (and slightly edited) by Klinton Bicknell

UC San Diego, Department of Linguistics

15 July 2009

Goals of this talk

- Briefly review generalized linear models and how to use them
- Give a precise description of multi-level models
- Show how to draw inferences using a multi-level model (*fitting* the model)
- Discuss how to interpret model parameter estimates
  - Fixed effects
  - Random effects

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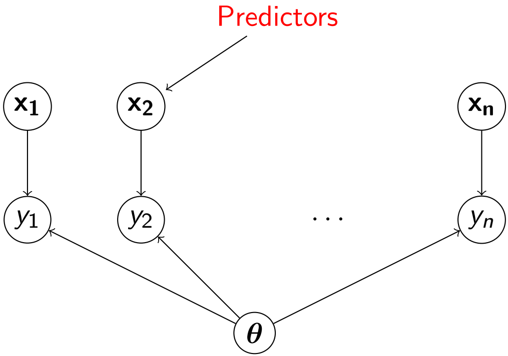

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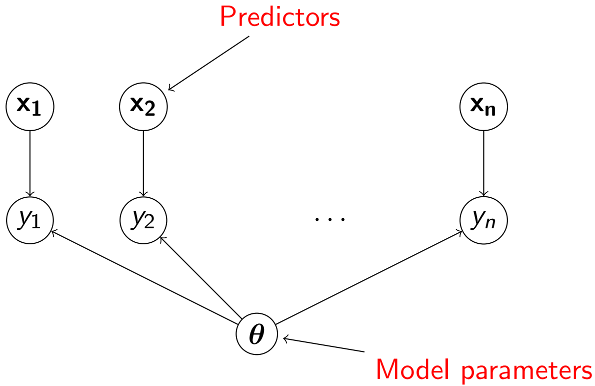

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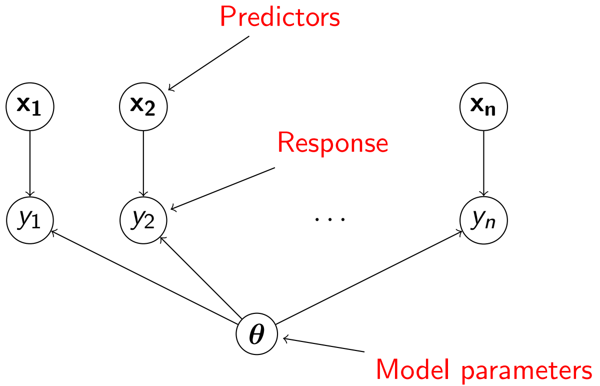

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4. There is some noise distribution of Y around the predicted mean μ of Y:

$$P(Y = y; \mu)$$

Reviewing GLMs III

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- This gives us the traditional linear regression equation:

$$Y = \overbrace{\alpha + \beta_1 X_1 + \cdots + \beta_n X_n}^{\text{Predicted Mean } \mu = \eta} + \overbrace{\epsilon}^{\text{Noise} \sim N(0,\sigma)}$$

Reviewing GLMs IV

$$Y = \overbrace{\alpha + \beta_1 X_1 + \cdots + \beta_n X_n}^{\text{Predicted Mean}} + \overbrace{\epsilon}^{\text{Noise} \sim N(0,\sigma)}$$

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- We need to draw inferences about α, β, and σ
- e.g., "Does neighborhood density affects RT?"→ is β reliably non-zero?

Reviewing GLMs VI

- We'll use length-4 nonword data from (Bicknell et al., 2008), such as:

Few neighbors			*Many neighbors*		
`gaty`	`peme`	`rixy`	`lish`	`pait`	`yine`

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Few neighbors			*Many neighbors*		
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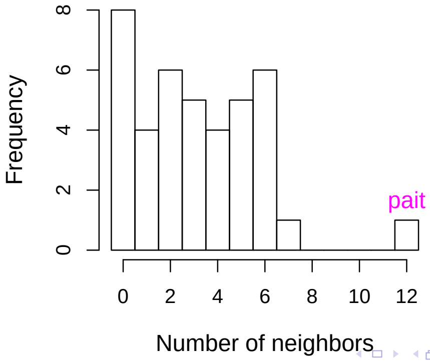
- There's a wide range of neighborhood density:



Reviewing GLMs VII: maximum-likelihood model fitting

$$RT_i = \alpha + \beta X_i + \overbrace{\epsilon_i}^{\sim N(0,\sigma)}$$

- Here's a translation of our simple model into R:

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RT ~ 1 + x
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- Example of fitting via maximum likelihood: one subject from Bicknell et al. (2008)

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> m <- glm(RT ~ neighbors, d, family="gaussian")
```

Gaussian noise, implicit intercept

```
> summary(m)
[...]
             Estimate Std. Error t value Pr(>|t|)
(Intercept)   382.997     26.837  14.271   <2e-16 ***
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$\widehat{\alpha}$ → Estimate (Intercept): 382.997

$\widehat{\beta}$ → Estimate neighbors: 4.828

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$\widehat{\alpha}$ → 382.997

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Reviewing GLMs: maximum-likelihood fitting VIII

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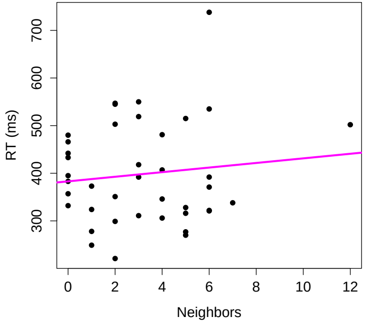

Reviewing GLMs IX: Bayesian model fitting

$$P(\{\beta_i\},\sigma|Y) = \frac{\overbrace{P(Y|\{\beta_i\},\sigma)}^{\text{Likelihood}}\ \overbrace{P(\{\beta_i\},\sigma)}^{\text{Prior}}}{P(Y)}$$

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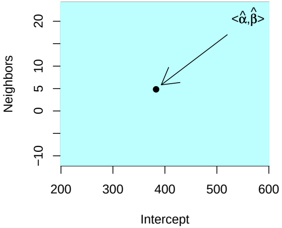

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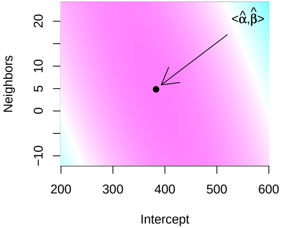

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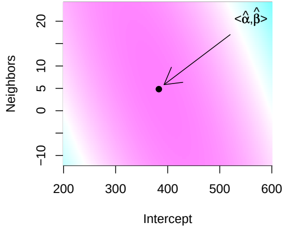

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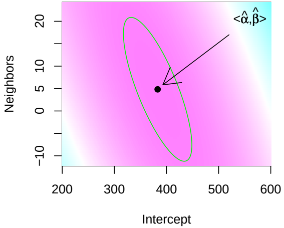
- Multiply by likelihood $\rightarrow$ posterior probability distribution over (α, β, σ)
- Bound the region of highest posterior probability containing 95% of probability density $\rightarrow$ HPD confidence region



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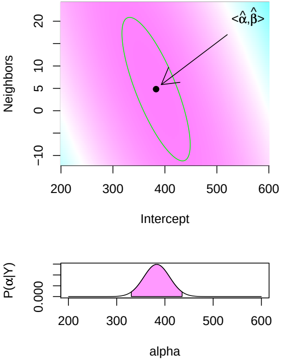

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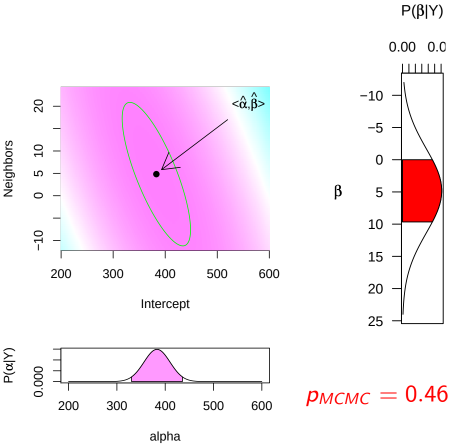


$p_{MCMC} = 0.46$

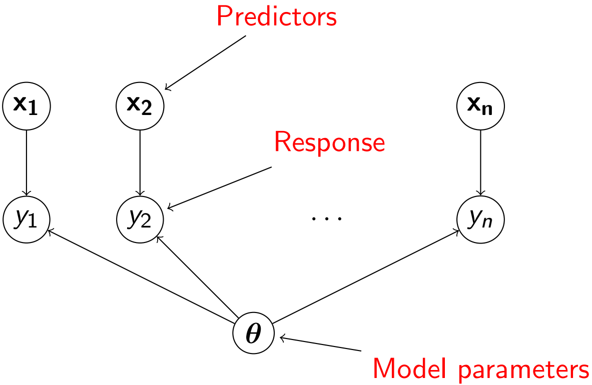
- p_{MCMC} (Baayen et al., 2008) is 1 minus the largest possible symmetric confidence interval wholly on one side of 0

Multi-level Models

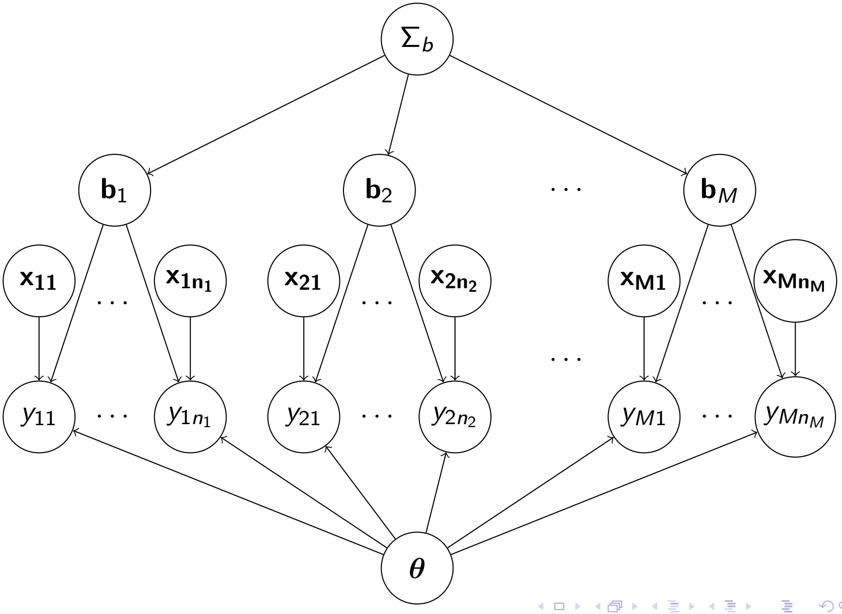
- But of course experiments don't have just one participant
- Different participants may have different idiosyncratic behavior
- And items may have idiosyncratic properties too
- We'd like to take these into account, and perhaps investigate them directly too.
- This is what multi-level (hierarchical, mixed-effects) models are for!

Multi-level Models II

- Recap of the graphical picture of a single-level model:



Multi-level Models III: the new graphical picture



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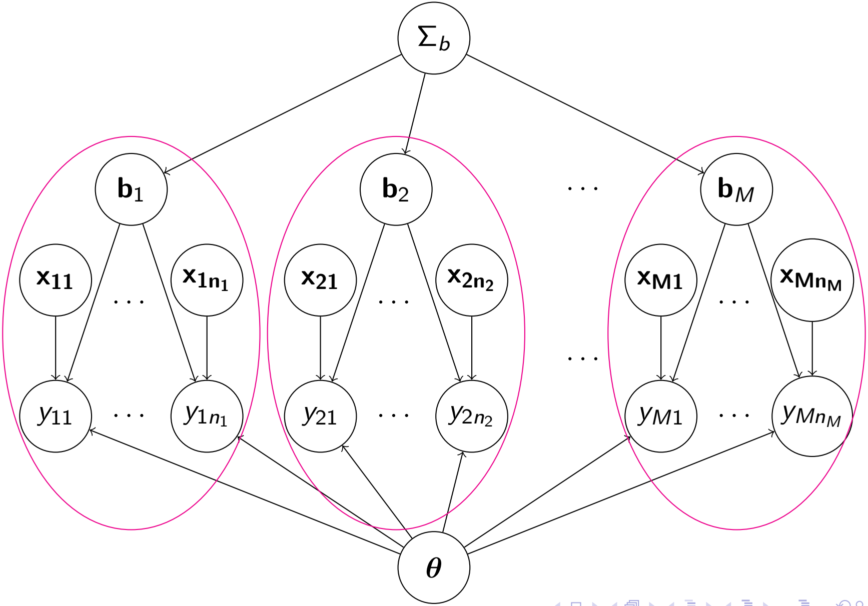

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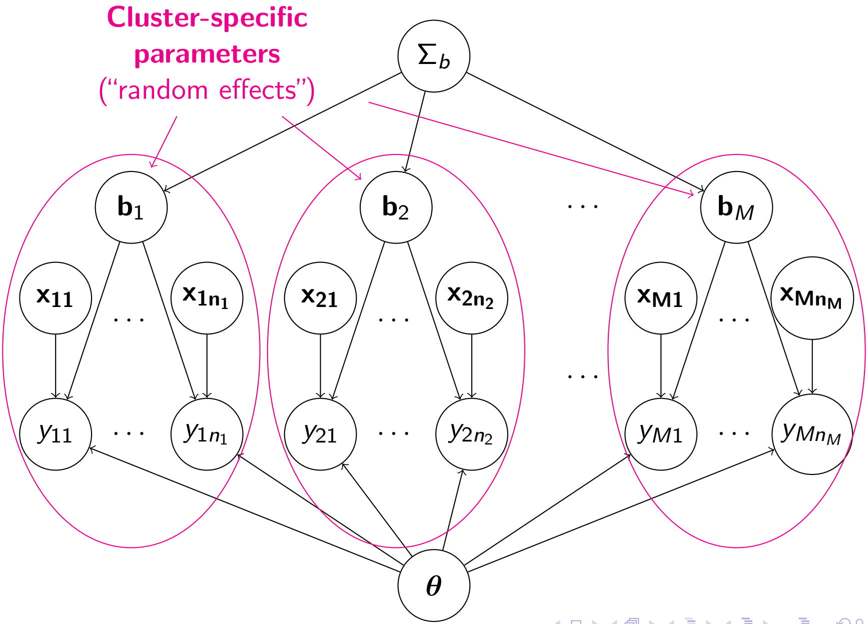

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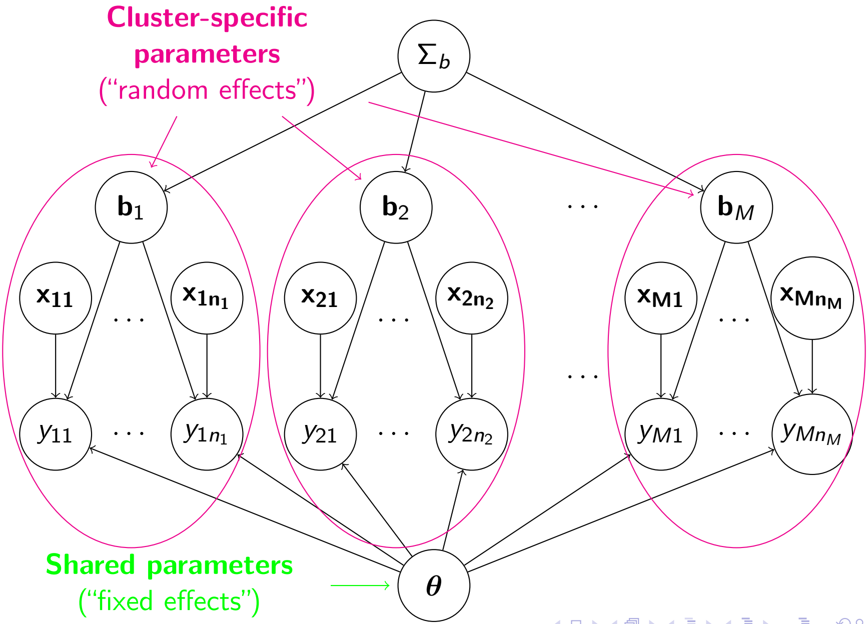

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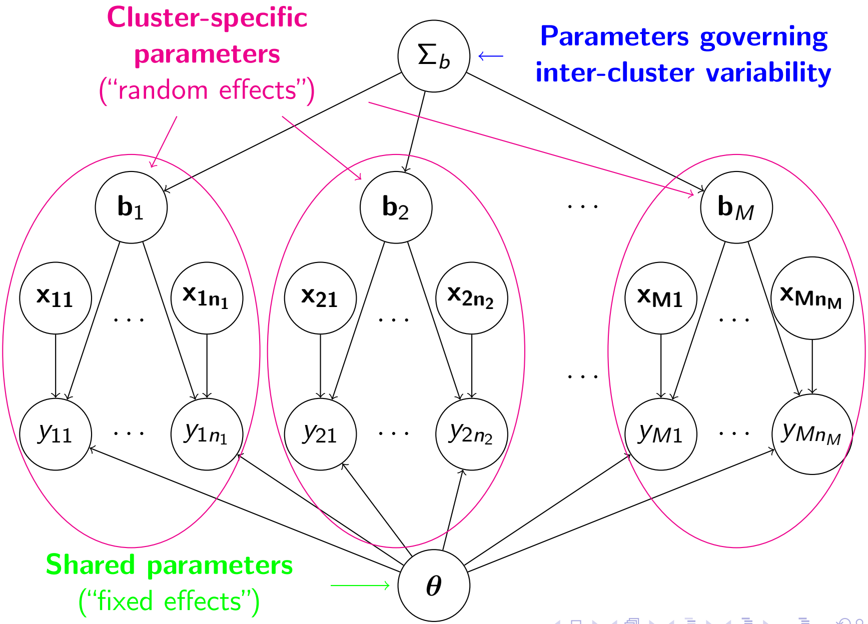

Multi-level Models IV

An example of a multi-level model:

- Back to your lexical-decision experiment

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- Non-words with different *neighborhood densities* should have different average decision time
- Additionally, different participants in your study may also have:
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Multi-level Models IV

An example of a multi-level model:

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- Additionally, different participants in your study may also have:
  - different overall decision speeds
  - differing sensitivity to neighborhood density
- You want to draw inferences about all these things at the same time

Multi-level Models V: Model construction

- Once again we'll assume for simplicity that the number of word neighbors x has a linear effect on mean reading time, and that trial-level noise is normally distributed*

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- This is invaluable for achieving deeper understanding of both your analysis and your data

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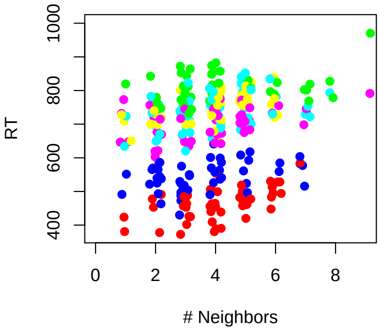
```
## simulate some data
> sigma.b <- 125        # inter-subject variation larger than
> sigma.e <- 40         # intra-subject, inter-trial variation
> alpha <- 500
> beta <- 12
> M <- 6                               # number of participants
> n <- 50                              # trials per participant
> b <- rnorm(M, 0, sigma.b)            # individual differences
> nneighbors <- rpois(M*n,3) + 1       # generate num. neighbors
> subj <- rep(1:M,n)
> RT <- alpha + beta * nneighbors + # simulate RTs!
    b[subj] + rnorm(M*n,0,sigma.e)  #
```

Multi-level Models VII: simulating data



- Participant-level clustering is easily visible

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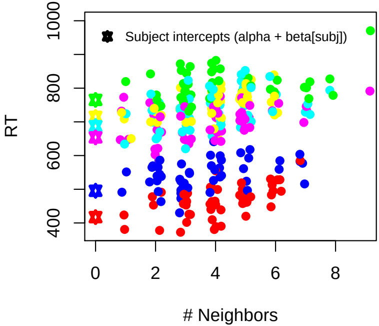

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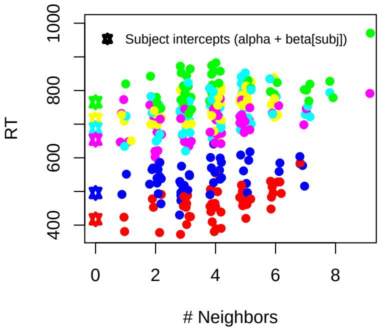

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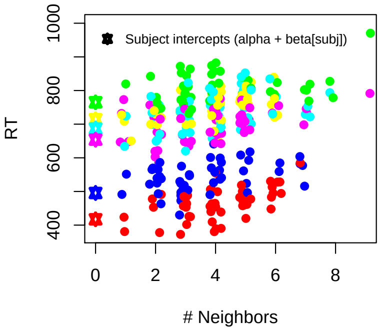

- Participant-level clustering is easily visible
- This reflects the fact that inter-participant variation (125ms) is larger than inter-trial variation (40ms)
- And the effects of neighborhood density are also visible

Statistical inference with multi-level models

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- We *have* data and we need to infer a model
  - Specifically, the "fixed-effect" parameters α, β, and σ_ϵ, plus the parameter governing inter-subject variation, σ_b
  - e.g., hypothesis tests about effects of neighborhood density: can we reliably infer that β is {non-zero, positive, . . . }?

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  - e.g., hypothesis tests about effects of neighborhood density: can we reliably infer that β is {non-zero, positive, ...}?
- Fortunately, we can use the same principles as before to do this:
  - The principle of maximum likelihood
  - Or Bayesian inference

Fitting a multi-level model using maximum likelihood

$$RT_{ij} = \alpha + \beta x_{ij} + \overbrace{b_i}^{\sim N(0,\sigma_b)} + \overbrace{\epsilon_{ij}}^{\text{Noise}\sim N(0,\sigma_\epsilon)}$$

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> m <- lmer(time ~ neighbors.centered +
    (1 | participant),dat,REML=F)
> print(m,corr=F)

[...]
Random effects:
 Groups      Name        Variance Std.Dev.
 participant (Intercept)  4924.9   70.177
 Residual                19240.5  138.710
Number of obs: 1760, groups: participant, 44

Fixed effects:
                   Estimate Std. Error t value
(Intercept)         583.787     11.082   52.68
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($\hat{\alpha}$ → 583.787)

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  - Variability in average RT in the population from which the participants were drawn has standard deviation 70.18ms

Inferences about cluster-level parameters

$$RT_{ij} = \alpha + \beta x_{ij} + \overbrace{b_i}^{\sim N(0,\sigma_b)} + \overbrace{\epsilon_{ij}}^{\text{Noise}\sim N(0,\sigma_\epsilon)}$$

- What about the participants' idiosyncracies themselves—the b_i?

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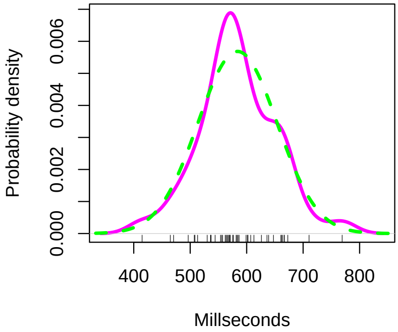
- The BLUPS are the conditional modes of b_i—the choices that maximize the above probability

Inferences about cluster-level parameters II

- The BLUP participant-specific 'base' RTs for this dataset are black lines on the base of this graph



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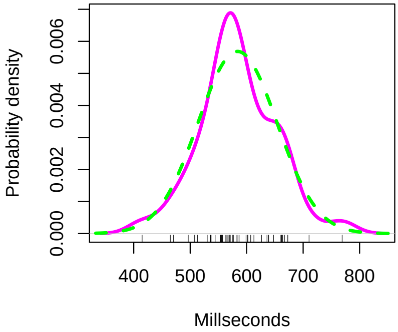

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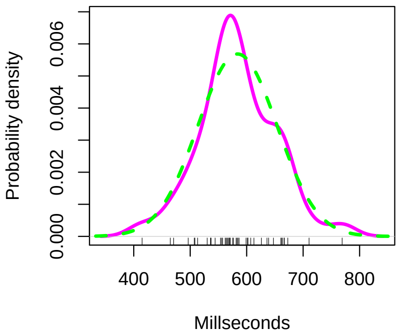

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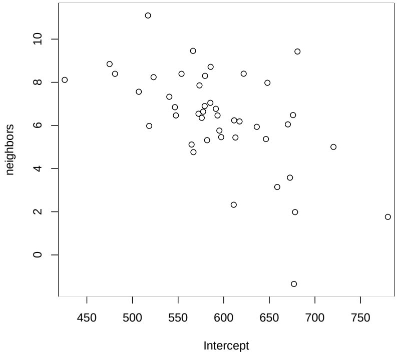
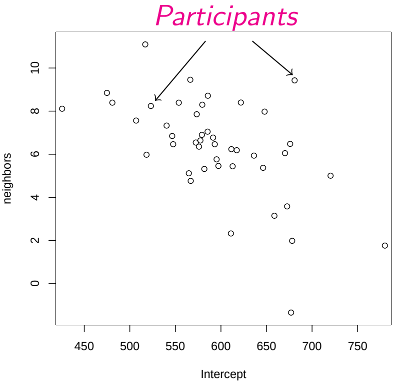
These three numbers jointly characterize $\widehat{\Sigma}_b$

Inference about cluster-level parameters IV



Inference about cluster-level parameters IV



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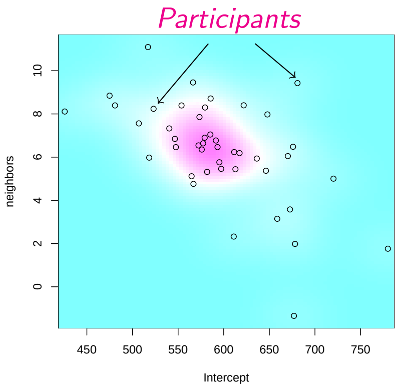

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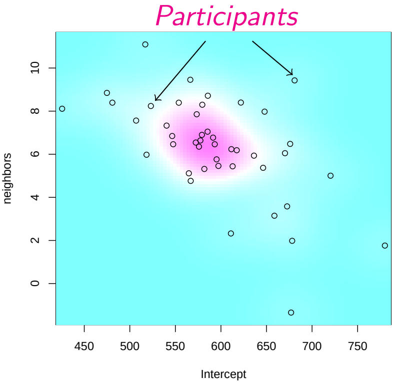

- Correlation visible in participant-specific BLUPs

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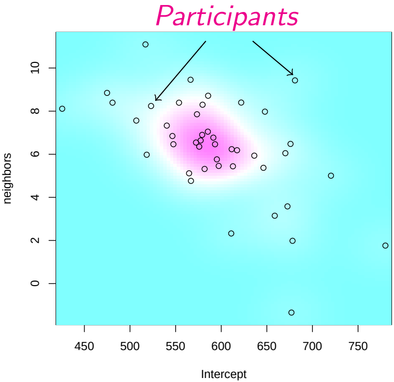

- Correlation visible in participant-specific BLUPs
- Participants who were faster overall also tend to be more affected by neighborhood density

$$\widehat{\Sigma} = \begin{pmatrix} 70.20 & -0.3097 \\ -0.3097 & 4.41 \end{pmatrix}$$

Bayesian inference for multilevel models

$$P(\{\beta_i\}, \sigma_b, \sigma_\epsilon | Y) = \frac{\overbrace{P(Y|\{\beta_i\}, \sigma_b, \sigma_\epsilon)}^{\text{Likelihood}} \overbrace{P(\{\beta_i\}, \sigma_b, \sigma_\epsilon)}^{\text{Prior}}}{P(Y)}$$

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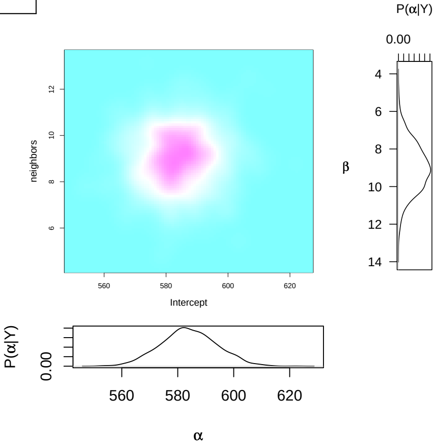

Why do you care???

- You may be asking yourself:

  Why did I come to this workshop? I could do everything you just did with an ANCOVA, treating participant as a random factor, or by looking at participant means.

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- Yes, but there are several respects in which multi-level models go beyond AN(C)OVA:

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Why did I come to this workshop? I could do everything you just did with an ANCOVA, treating participant as a random factor, or by looking at participant means.

- Yes, but there are several respects in which multi-level models go beyond AN(C)OVA:
    1. They handle *imbalanced datasets* just as well as balanced datasets

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Summary

- Multi-level models may seem strange and foreign
- But all you really need to understand them is three basic things
    - Generalized linear models
    - The principle of maximum likelihood
    - Bayesian inference
- As you will see in the rest of the workshop, these models open up many new interesting doors!

References I

Agresti, A. (2002). *Categorical Data Analysis*. Wiley, second edition.

Baayen, R. H., Davidson, D. J., and Bates, D. M. (2008). Mixed-effects modeling with crossed random effects for subjects and items. *Journal of Memory and Language*. In press.

Bicknell, K., Elman, J. L., Hare, M., McRae, K., and Kutas, M. (2008). Online expectations for verbal arguments conditional on event knowledge. In *Proceedings of the 30th Annual Conference of the Cognitive Science Society*, pages 2220–2225.

Jaeger, T. F. (2008). Categorical data analysis: Away from ANOVAs (transformation or not) and towards logit mixed models. *Journal of Memory and Language*. In press.